

Charge Screening and Confinement in Hot 3-D QEDG. Grignani^{1a}, G. Semenoff^{2b}, P. Sodano^{3a} and O. Tirkkonen^{4b}

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Abstract

We examine the possibility of a confinement-deconfinement phase transition at finite temperature in both parity invariant and topologically massive three-dimensional quantum electrodynamics. We review an argument showing that the Abelian version of the Polyakov loop operator is an order parameter for confinement, even in the presence of dynamical electrons. We show that, in the parity invariant case, where the tree-level Coulomb potential is logarithmic, there is a Berezinskii-Kosterlitz-Thouless transition at a critical temperature ($T_c = e^2/8\pi + \mathcal{O}(e^4/m)$, when the ratio of the electromagnetic coupling and the temperature to the electron mass is small). Above T_c the electric charge is not confined and the system is in a Debye plasma phase, whereas below T_c the electric charges are confined by a logarithmic Coulomb potential, qualitatively described by the tree-level interaction. When there is a topological mass, no matter how small, in a strict sense the theory is not confining at any temperature; the model exhibits a screening phase, analogous to that found in the Schwinger model and two-dimensional QCD with massless adjoint matter. However, if the topological mass is much smaller than the other dimensional parameters, there is a temperature for which the range of the Coulomb interaction changes from the inverse topological mass to the inverse electron mass. We speculate that this is a vestige of the BKT transition of the parity-invariant system, separating regions with screening and deconfining behavior.

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1 Introduction

One of the most intriguing features of a gauge theory is the possibility of confinement. In a confining system, there are no “in” or “out” fields appearing in the asymptotic states which have color charges [1, 2]. As a result, all asymptotic states are singlets under the symmetry transformations in the color group. This generally occurs in one of two ways. First, as it is widely believed to be the case in four-dimensional quantum chromodynamics (QCD), the charged fields appearing in the bare action - quarks and gluons - are permanently confined into color singlet bound states - mesons and baryons - which make up the entire spectrum of asymptotic states. Second, as well as the appearance of color singlet bound states, it is possible that the charges of bare fields are completely screened, so that they interpolate physical fields which occur in the spectrum but create color singlet states. This possibility has been raised for the electron field in the Schwinger model [3] and in two dimensional adjoint QCD where the bare quark mass is zero [4]. This situation is often referred to as “screening” rather than confinement.

At finite temperature, the difference between a confined and deconfined phase is less evident than at zero temperature. There is no concept of asymptotic states and the quantitative observable features are thermodynamic variables and correlation functions governing the propagation of external influences. The commonly used test for confinement in a gauge theory at finite temperature is its ability to screen static external charges. The operator which probes electric screening is the Polyakov loop operator [5, 6] which is the trace of the path ordered exponential of the gauge field on a path which links the periodic Euclidean time ⁵

$$P(\vec{x}) = \text{tr} \mathcal{P} e^{i \int_0^{1/T} d\tau A_0(\tau, \vec{x})} \quad (1)$$

The expectation value of this operator is the exponential of the free energy, $F(\vec{x})$, which is required to immerse a classical fundamental representation quark source in the gauge field medium at the point \vec{x} ,

$$\langle P(\vec{x}) \rangle = e^{-F(\vec{x})/T} \quad (2)$$

If the expectation value is zero, corresponding to infinite free energy, this is interpreted as a signal of confinement. In this case, the system is not capable of screening the electric flux which is necessarily created with the external quark source and the electric flux takes up a configuration which has infinite energy. It is important that the infinite energy arises from an infrared, rather than ultraviolet divergence since the latter occurs even in non-confining theories and could be cured by introducing a fundamental cutoff.

If the expectation value of the Polyakov loop is non-zero, and the free energy finite, this is interpreted as the system being in a deconfined phase. The electric flux associated with the source is screened by the medium.

In compact gauge theories, the Polyakov loop operator can be used as an order parameter for confinement in either pure Yang-Mills gluo-dynamics or

⁵We use units where Planck’s constant, the speed of light and Boltzmann’s constant are one. For a discussion of the path integral formulation of finite temperature gauge theory see [7].

in a matter-coupled gauge theory when the matter is in either the adjoint representation, or some other representation whose degrees of freedom transform trivially under the center of the gauge group. In these cases, there is an invariance of the finite temperature path integral under gauge transformations which twist by an element of the center of the gauge group in the periodic Euclidean time. The operators in the action are all invariant and remain periodic (or anti-periodic in the case of fermions) under such a twisted gauge transform, but the operator $P(\vec{x})$ is transformed to $ZP(\vec{x})$ where Z is the central element. Thus, the finite temperature theory has an effective global symmetry whose transformations are the cosets of the set of all gauge transformations under which the path integral is symmetric modulo those which are strictly periodic in Euclidean time. Whether this symmetry is broken or not is a well defined question. Spontaneous breaking of the symmetry is related to deconfinement [5, 6] and $P(\vec{x})$ is an order parameter. This order parameter has been particularly useful in characterizing the nature [8]-[11] and quantitative aspects [12]-[14] of the confinement-deconfinement phase transition in a wide array of pure gauge theories.

When the gauge theory is coupled to matter fields which transform non-trivially under the center of the group, the symmetry is broken explicitly and its realization can no longer be used as a probe for confinement. An example is QCD with quarks in the fundamental representation of $SU(3)$. In that case, the question of distinguishing a confining and non-confining phase of the finite temperature gauge theory is more sophisticated [15].

Recently it has been argued [16, 17] that the Abelian analog of the Polyakov loop operator,

$$P_{\tilde{e}}(\vec{x}) \equiv e^{i\tilde{e} \int_0^{1/T} d\tau A_0(\tau, \vec{x})} \quad (3)$$

could be used as an order parameter for confinement in Abelian gauge theories, even in the presence of dynamical charged particles. The requirements are that the dynamical charged fields in the gauge theory must have charges which are integral multiples of some basic charge, which we denote by e . Also, a technical requirement is that the charged fields have a mass gap and that the field theory has a finite ultraviolet cutoff. In the limit where the cutoff is removed, it is usually necessary to define the loop operator by multiplicative renormalization. In order that (3) be an order parameter, the charge \tilde{e} appearing there **must not be an integer multiple of the basic charge**: $\tilde{e} \neq e \cdot \text{integer}$.

The expectation value of the operator (3) measures the response of an electrodynamic system to placing a classical *incommensurate* charge \tilde{e} at point \vec{x} . The quantity

$$F_{\tilde{e}}(\vec{x}) = -T \ln \langle P_{\tilde{e}}(\vec{x}) \rangle \quad (4)$$

is the free energy of the system in the presence of the classical charge (minus the free energy when the charge is absent). If this free energy is finite, the system is not confining. If the free energy is infinite, this implies that the expectation value (3) must be zero. This means that it takes an infinite amount of energy to immerse a classical charge in the system, implying that it is in a confining phase. If the charge in the loop operator were a commensurate, rather than incommensurate one, its electric field could be screened by producing a finite number of dynamical charged particle-antiparticle pairs, using the appropriate number of particles or antiparticles to screen the external charge

and allowing the remaining dynamical particles to escape to infinity. This process would take a finite amount of energy. For this reason, we expect that the loop operator with a commensurate charge would always have a non-zero expectation value. At zero temperature, the pair production would take a threshold energy of the mass of the particles produced. At finite temperature the thermally activated particles are already present so that screening of this sort takes a small amount of energy.

Further information can be obtained from the correlators,

$$F_{\tilde{e}_1 \dots \tilde{e}_n}(\vec{x}_1, \dots, \vec{x}_n) = -T \ln \left\langle \prod_{i=1}^n P_{\tilde{e}_i}(\vec{x}_i) \right\rangle \quad (5)$$

which give the electrostatic energy of an array of charges \tilde{e}_i situated at points \vec{x}_i , respectively. For example, the two-point correlator gives the effective potential between a positive and negative charge,

$$V_{\tilde{e}, -\tilde{e}}(\vec{x}, \vec{y}) \equiv F_{\tilde{e}, -\tilde{e}}(\vec{x}, \vec{y}) = -T \ln \langle P_{\tilde{e}}(\vec{x}) P_{-\tilde{e}}(\vec{y}) \rangle \quad (6)$$

If the clustering property of this expectation value holds, vanishing or non-vanishing of the expectation value of a single loop operator is related to the asymptotic behaviour of the potential: if

$$\lim_{|\vec{x}-\vec{y}| \rightarrow \infty} V_{\tilde{e}, -\tilde{e}}(\vec{x}, \vec{y}) = \infty \quad (7)$$

the two point function has the behavior

$$\lim_{|x-y| \rightarrow \infty} \langle P_{\tilde{e}}(\vec{x}) P_{-\tilde{e}}(\vec{y}) \rangle = 0 \quad (8)$$

and the cluster decomposition property implies that the expectation value of the single operator should vanish

$$\langle P_{\tilde{e}}(\vec{x}) \rangle = 0 \quad (9)$$

This characterizes confinement. If, on the other hand

$$\lim_{|\vec{x}-\vec{y}| \rightarrow \infty} V_{\tilde{e}, -\tilde{e}}(\vec{x}, \vec{y}) = \text{finite constant} \quad (10)$$

the potential is not confining, the two-point correlator has the behavior

$$\lim_{|x-y| \rightarrow \infty} \langle P_{\tilde{e}}(\vec{x}) P_{-\tilde{e}}(\vec{y}) \rangle = \text{constant} , \quad (11)$$

clustering implies that

$$\langle P_{\tilde{e}}(\vec{x}) \rangle = \text{constant} \quad (12)$$

and an isolated external incommensurate charge has finite energy.

The expectation value of the loop operator in (3) is governed by a particular discrete global symmetry, Z , isomorphic to the additive group of the integers, which appears in the euclidean path integral at finite temperature. To see its origin, consider the euclidean finite temperature functional integral expression for the partition function [7],

$$Z[T] = \int \prod_{\vec{x}, \tau \in [0, 1/T]} [dA_\mu(\tau, \vec{x}) d\psi(\tau, \vec{x}) d\bar{\psi}(\tau, \vec{x})] e^{-S[A, \psi, \bar{\psi}]} \quad (13)$$

where the action is

$$S[A, \psi, \bar{\psi}] = \int_0^{1/T} d\tau \int d\vec{x} \left(\frac{1}{4} F_{\mu\nu}^2(\tau, \vec{x}) + \bar{\psi}(\tau, \vec{x}) (\gamma_\mu D_\mu + m) \psi(\tau, \vec{x}) \right) \quad (14)$$

with $D_\mu = \partial_\mu - ieA_\mu$ and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. The boundary conditions in time are periodic for the photon,

$$A_\mu(\tau = 1/T, \vec{x}) = A_\mu(0, \vec{x}) \quad (15)$$

and antiperiodic for the electron,

$$\psi(\tau = 1/T, \vec{x}) = -\psi(\tau = 0, \vec{x}) \quad , \quad \bar{\psi}(\tau = 1/T, \vec{x}) = -\bar{\psi}(\tau = 0, \vec{x}) \quad (16)$$

The path integral (13) is symmetric under gauge transformations,

$$A'_\mu(\tau, \vec{x}) = A_\mu(\tau, \vec{x}) + \nabla_\mu \chi(\tau, \vec{x}) \quad (17)$$

$$\psi'(\tau, \vec{x}) = e^{ie\chi(\tau, \vec{x})} \psi(\tau, \vec{x}) \quad , \quad \bar{\psi}'(\tau, \vec{x}) = \bar{\psi}(\tau, \vec{x}) e^{-ie\chi(\tau, \vec{x})} \quad (18)$$

when the gauge transformation function $\chi(\tau, \vec{x})$ has periodic derivatives,

$$\nabla_\mu \chi(\tau = 1/T, \vec{x}) = \nabla_\mu \chi(\tau = 0, \vec{x}) \quad (19)$$

and it is periodic up to an integer multiple of $2\pi/e$,

$$\chi(\tau = 1/T, \vec{x}) = \chi(\tau = 0, \vec{x}) + 2\pi n/e \quad , \quad n \in \mathbb{Z} \quad (20)$$

The group of all gauge transformations modulo those which are strictly periodic is \mathbb{Z} , the additive group of the integers. This is a global symmetry. The Polyakov loop operator transforms non-trivially under the coset when its charge is not an integer multiple of the electron charge,

$$P'_e(\vec{x}) = P_e(\vec{x}) \cdot e^{2\pi i n \tilde{e}/e} \quad (21)$$

It can therefore be used as an operator to explore the realization of \mathbb{Z} in the statistical model specified by the path integral (13). If the symmetry is unbroken, the loop operator averages to zero and the system is in the confining phase. If it is spontaneously broken, the loop operator can have a non-zero expectation value. The system is then in a non-confining phase. We shall discuss two such phases. One is the Debye plasma phase which is characterized by the exponential decay in the asymptotic behaviour of the electric two-particle potential, related to the Debye screening length of the plasma. It is also related to the mass of elementary excitations in a lower dimensional theory [18]. The other is what is termed a “screening phase”. This phase also has an electric screening length, as well as magnetic screening. The main difference between these two phases is in the temperature dependence of the screening length. In the screening phase, the screening length persists at zero temperature, whereas the Debye mass vanishes at zero temperature.

The \mathbb{Z} symmetry has a physical interpretation in terms of the charge of physical states [19]. In path integral quantization, as we shall discuss in Section III, the temporal component of the gauge field A_0 arises as a Lagrange

multiplier to enforce gauge invariance. The projection operator which guarantees gauge invariance in the construction of the path integral is obtained by exponentiating the generator of infinitesimal gauge transformations and integrating over all gauge transformations,

$$\mathcal{P} = \frac{1}{\text{const.}} \int [dA_0(\vec{x})] e^{\frac{i}{T} \int d\vec{x} (\vec{\nabla} A_0 \cdot \vec{E} - A_0 e : \psi^\dagger \psi :)} \quad (22)$$

and the density matrix for the Gibbs ensemble is

$$\rho = \mathcal{P} \frac{e^{-H/T}}{Z[T]} \quad (23)$$

The Z transformation

$$A_0(\vec{x}) \rightarrow A_0(\vec{x}) + 2\pi n T / e \quad (24)$$

results in

$$\mathcal{P} \rightarrow \mathcal{P} \cdot e^{2\pi i n Q / e} \quad (25)$$

where

$$Q \equiv e \int d\vec{x} : \psi^\dagger(\vec{x}) \psi(\vec{x}) : \quad (26)$$

is the (normal ordered) electric charge operator. If the Z symmetry is not broken, all physical states with non-zero weight in the thermal ensemble have quantized charges,

$$e^{2\pi i n Q / e} \cdot \mathcal{P} \frac{e^{-H/T}}{Z[T]} = \mathcal{P} \frac{e^{-H/T}}{Z[T]} \quad (27)$$

If the Z symmetry is broken, there are states contributing to the thermal ensemble which have non-quantized charges. Intuitively, these can occur in a deconfined theory since the long-ranged electric fields accompanying arbitrarily diffuse charge distributions would have finite energy. Such states would not be allowed in the confining phase. Note that, in this operator picture, the Z symmetry is not a symmetry transform of the density matrix in the usual sense. In fact the existence of this symmetry is related only to the question of whether $\exp(2\pi i Q / e)$ is the unit operator when operating on the states that have non-zero weight in the density matrix.

At $T = 0$, and for the physical value of the electromagnetic coupling constant, 3+1-dimensional electrodynamics does not exhibit a confining phase. It is in the deconfined Coulomb phase at zero temperature and forms a Debye plasma at finite temperature and density. There is a conjecture that if the electric charge of QED could be increased to some critical value, there would be a phase transition to a chiral symmetry breaking and confining phase [20]. It is reasonable to expect that the resulting strongly coupled system would have a confinement-deconfinement transition at some finite temperature.

QED in 1+1 dimensions with a massive electron, i.e. the massive Schwinger model, is confining and the Z symmetry is not broken. In the case of the massless Schwinger model, the Z symmetry is spontaneously broken [21, 19]. This spontaneous breaking of Z is interpreted as screening, rather than deconfinement. A similar situation appears in 1+1-dimensional QCD with massless quarks [22, 4]. With massive quarks, the question of confinement and the

confinement-deconfinement transition in the large N limit of 1+1-dimensional QCD both at finite temperature [23, 24, 25] and at zero temperature [26] has been examined by several authors.

When the electron has a mass, parity invariant quantum electrodynamics (QED) in 2+1 dimensions is believed to be a confining theory. The tree level Coulomb interaction varies logarithmically with distance. Its entire spectrum is bound states, although the bound states can have arbitrarily large sizes. The perturbative self-energy of the electron has a logarithmic infrared infinity [27]. Also, when the number of electron flavors is small enough, confinement, accompanied by chiral symmetry breaking is believed to persist in the limit as the bare electron mass is put to zero [28]. In a previous paper by three of us it was argued that 2+1-dimensional QED has a phase transition from the confined to a deconfined phase at a critical temperature [17]. The phase transition is of Berezinsky-Kosterlitz-Thouless [29, 30] (BKT) type with non-universal, temperature dependent critical exponents. This is the standard phase transition of the Coulomb gas, which is a reasonable characterization of the thermal state of QED in the limit where the density of thermally excited particles is low. This is the case when the mass of the particles is greater than the temperature. The Polyakov loop operators have power-law correlators in the confining phase. In the deconfined phase there is an electric, Debye mass which makes the Coulomb interaction short-ranged. There is a universal quantity associated with the BKT phase transition, the bulk modulus of spin waves in the massless phase. This predicts the power law behavior of Polyakov loop correlators in the confined theory.

It is interesting to ask what happens in the case where the three dimensional gauge theory is not parity invariant, but the action contains a topological mass term for the photon. Of course, one would expect that the resulting photon mass [31, 32] provides an infrared cutoff and thereby removes the long-ranged interactions that are responsible for confinement. However, if the topological mass is very small, so that the confinement scale is much larger than the photon mass, we expect that some effects of confinement persist at short distances.

To understand what effect topological mass has on the Z symmetry, consider the Chern-Simons (CS) action in Euclidean space

$$S_{CS} = i\frac{\kappa}{2} \int_0^{1/T} d\tau \int d\vec{x} \epsilon_{0ij} (2A_0 \nabla_i A_j - A_i \dot{A}_j) \quad (28)$$

The gauge transformation

$$A_0(\tau, \vec{x}) \rightarrow A_0(\tau, \vec{x}) + \frac{d}{d\tau} \left(\frac{e}{\kappa} n T \tau \right) \quad (29)$$

where e is the basic unit of charge of all matter fields, results in the change of the action

$$S_{CS} \rightarrow S_{CS} + i n e g \quad (30)$$

where g is the magnetic charge,

$$g = \int d\vec{x} \epsilon_{0ij} \nabla_i A_j \quad (31)$$

When, as it happens on an open space such as \mathcal{R}^2 which we consider here, the magnetic charge is not quantized, the Chern-Simons action is not invariant

under the gauge transformation unless $n = 0$. The presence of a Chern-Simons term in the action then would break the Z symmetry explicitly.

On the other hand, if the space is compact, or if we impose boundary conditions on an open space, such as the plane, so that the gauge field can be stereographically projected onto a compact space, the magnetic charge should obey the Dirac quantization condition,

$$eg = 2\pi k \quad (32)$$

where k is an integer and e is the basic unit of charge of the matter fields. In this case, the Chern-Simons term is invariant under gauge transformations with non-zero winding number. The global symmetry is Z , even in the absence of matter.

The Polyakov loop operator for a basic charge (remember that, because of the presence of monopoles, the charge is quantized) transforms as

$$e^{ie \int_0^{1/T} d\tau A_0(\tau, \vec{x})} \rightarrow e^{ie^2/\kappa} \cdot e^{ie \int_0^{1/T} d\tau A_0(\tau, \vec{x})} \quad (33)$$

We denote the coefficient of the Chern-Simons term as

$$\kappa = \frac{e^2}{2\pi} \frac{p}{q}, \quad (34)$$

so that the Polyakov loop operator transforms by the phase $e^{2\pi i q/p}$. Depending of the ranges of p and q , there are three possibilities:

- If p/q is an irrational number, then the symmetry is Z . In finite volume this will imply that all charges are confined and only neutral configurations are allowed.
- If p and q are integers so that p/q is rational then the symmetry is the finite cyclic group Z_p . In this case, charges which are not integral multiples of p are confined.
- If q is an integer and $p = 1$, there is no symmetry.

The latter condition is compatible with Gauss' law which, as we shall see in Section III, relates the total charge and magnetic flux of a quantum state on a compact space as

$$\frac{e^2}{2\pi} \frac{p}{q} g + Q = 0 \quad (35)$$

Since $g = \frac{2\pi}{e} \cdot \text{integer}$ and $Q = \text{integer} \cdot e$, it is necessary that p and q are integers. Furthermore, the basic state in this system has electric charge $p \cdot e$ and magnetic charge $q \cdot \frac{2\pi}{e}$.

In fact, we could think of the effective symmetry of the Euclidean path integral as just enforcing the global constraints on the charges which are contained in Gauss' law. We shall examine the question of whether this symmetry survives in the infinite volume limit in Section III.

In order to analyze symmetry breaking, the properties of correlators and other dynamical questions in parity invariant 2+1-dimensional QED, we shall compute the effective action for the Polyakov loop operator in Section II. This

method was advocated in the seminal work Svetitsky and Yaffe in the context of lattice gauge theories [11]. One first fixes the static temporal gauge,

$$\frac{d}{d\tau}A_0(\tau, \vec{x}) = 0 \quad (36)$$

and integrates all the degrees of freedom of the gauge theory except for A_0 which is associated with the Polyakov loop. This generates an effective theory for the order parameter which, for an initial $d+1$ -dimensional gauge theory is a d -dimensional scalar field theory with variable $A_0(\vec{x})$. This theory exhibits the global Z symmetry explicitly. Since, at finite temperature, fermions always have a mass gap, integrating out all the other degrees of freedom in the original theory produces only short-ranged interactions in the effective theory. Consequently, the critical behavior of $d+1$ -dimensional finite temperature QED is that of the d -dimensional effective local field theory. By studying the effective action, we are able to characterize the type of the phase transition and discuss the associated critical behavior.

In Section III we shall consider 2+1-dimensional pure QED with a Chern-Simons term in the presence of an array of external charges. We consider the case where the space is compact and discuss some boundary problems related to the presence of the Chern-Simons term on a compact surface. We discuss canonical quantization and the construction of the functional integral representation of the partition function $Z[T]$. We then find the exact effective action for A_0 by integrating all the spatial components of the gauge field. We use this effective action to study the realization of Z symmetry in the infinite volume limit.

In Section IV, we use a variational method to examine the behavior of both parity invariant and topologically massive QED in infinite volume. We confirm the existence of the BKT phase transition in the parity invariant theory. We also find indications of a similar transition in topologically massive QED. We speculate that the latter transition, although not strictly speaking a phase transition, separates two regions of the system with distinct physical behavior: a low temperature screening phase where there are charged particles bound into neutral bound states, as well as free neutral particles, and a high temperature deconfined phase where the bound states are absent and the charges of the particles are Debye-screened. Section V is devoted to a discussion of our results.

In this paper, we do not address the interesting and controversial question of whether domain walls exist between regions with different “orientations” of broken Z symmetry. The existence of these in non-Abelian gauge theories has been a subject of much discussion [33, 34, 16, 35]. If they did exist in QED, they would be very interesting and perhaps observable objects. Detailed analysis of this possibility is still an important problem.

It is interesting that in this paper we find a phase transition which is accessible to perturbation theory. This is a property of the critical line of BKT transitions; one end of the line is in a perturbative regime. To our knowledge, this is the only situation where one can study a confinement-deconfinement transition without the aid of numerical simulations. On the other hand, confirming the existence and properties of the transition which we discuss by numerical simulations would be a most worthwhile project.

2 Effective action in parity invariant QED

We shall consider QED in 2+1 dimensions. As it is well known, in 2+1 dimensions, the minimal, two-component Dirac fermions violate parity [31, 32], so that if included in the action, they can generate a parity violating topological mass for the photon by radiative corrections [36, 37]. In this section we shall study the case where the electron has mass but the photon is massless. For this purpose, we shall use parity invariant four-component fermions which are obtained by dimensional reduction of the 3+1-dimensional Dirac operator. The resulting model has two species of massive two-component fermions where the mass terms have opposite signs. The Euclidean action is

$$S = \int d^3x \left[\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(\gamma \cdot (\nabla - ieA) + m)\psi \right] \quad (37)$$

We shall find the effective action for the Polyakov loop operator (3). At finite temperature, it is possible to use a gauge transformation to set the temporal component of the gauge field, $A_0(\tau, \vec{x})$, independent of the Euclidean time, τ . In that gauge,

$$P_{\vec{e}}(\vec{x}) = e^{i\vec{e}A_0(\vec{x})/T} \quad (38)$$

and the effective field theory for this operator is the effective action for the static field $A_0(\vec{x})$. This two-dimensional effective field theory is obtained by integrating the other degrees of freedom from the path integral

$$e^{-S_{\text{eff}}[A_0]} \equiv \int [d\vec{A} d\psi d\bar{\psi}] e^{-S[A_\mu, \psi, \bar{\psi}]} \quad (39)$$

(The functional integral defines the effective action only modulo a temperature and volume dependent but A_0 independent constant.) The integral over the Grassman-valued Fermi fields yields the determinant of the Dirac operator, and

$$S_{\text{eff}}[A_0] = \int d\vec{x} \frac{\vec{\nabla} A_0 \cdot \vec{\nabla} A_0}{2T} - \ln \int [d\vec{A}] \det[\gamma \cdot (\nabla - ieA) + m] e^{-\int_0^{1/T} d\tau d\vec{x} F_{ij}^2/4} \quad (40)$$

The first term in (40) is the tree level action and the second term contains all quantum corrections. The remaining functional integral requires additional gauge fixing. The Z symmetry is a periodicity of the effective action under the field translation

$$S_{\text{eff}}[A_0] \rightarrow S_{\text{eff}}[A_0 + 2\pi nT/e] \quad (41)$$

The effective action, $S_{\text{eff}}[A_0]$, is non-local and non-polynomial. It can only be regarded as a local field theory when the momenta of interest are much smaller than the mass gaps of the fields which have been eliminated. In that case the effective action has a local expansion in powers of derivatives divided by masses.

At finite temperature the action (37) contains three parameters with the dimension of mass, the electron mass m , the gauge coupling e^2 , and temperature T . The loop expansion is super-renormalizable [31, 38]. In fact, with a gauge invariant and Euclidean Lorentz invariant regularization, it has no divergences whatsoever. This means that the effective theory that we obtain

does not contain an ultraviolet cutoff, and its *only* dimensional parameters are e^2 , m and T .

The dimensionless parameter which governs the accuracy of the loop expansion is the smaller of e^2/m and e^2/T . Also, in order for the local, derivative expansion of the effective action to be valid, it is necessary that either the electron mass or the temperature be larger than the momentum scales of interest. Then, the larger of m or T acts as the ultraviolet cutoff of the effective field theory.

At the tree-level, where the effective action is approximated by the first term on the right-hand-side of (40) only, we can easily compute the correlator of Polyakov loop operators,

$$\left\langle e^{i\vec{e}A_0(\vec{x})} e^{-i\vec{e}A_0(0)} \right\rangle_{\text{tree}} = \text{const.} \prod_{i < j} |\vec{x}|^{-\tilde{e}^2/4\pi T} \quad (42)$$

It has a scale invariant form with a temperature dependent exponent reminiscent of the correlators in the Gaussian spin-wave theory [39, 40]. This is a result of the marginally confining nature of the logarithmic Coulomb interaction. The behavior is between that of a confining theory where the correlator would exhibit the clustering property and decay exponentially at large distances and the deconfined theory where it would approach a constant. It is interesting to ask how quantum fluctuations would modify this result. We shall argue in the following that, at low temperatures the behavior (42) is qualitatively, though not quantitatively correct. At high temperatures, the correlator approaches a constant at large distances and the Z symmetry is broken.

We shall compute the effective action for $A_0(\vec{x})$ in the one-loop approximation and in an expansion in powers of derivatives of $A_0(\vec{x})$. The leading terms are

$$S_{\text{eff}}[A_0] = \int d^2x \left(Z(m, eA_0/T) \frac{1}{2T} \vec{\nabla} A_0 \cdot \vec{\nabla} A_0 - V(m, eA_0/T) \right) . \quad (43)$$

Here V is the effective potential for A_0 . $Z - 1$ arises from the expansion of the temporal components of the vacuum polarization function to linear order in $-\vec{\nabla}^2$.

$$\Pi_{00}(\omega = 0, \vec{k}^2) = \Pi_{00}(0, 0) + \vec{k}^2(Z - 1) + \dots \quad (44)$$

To one-loop order, the effective potential, $V(m, eA_0/T)$, is obtained from the fermion determinant in the constant background A_0 ,

$$V(m, eA_0/T) = \frac{1}{(\text{Vol.})} \log \det((-i\partial_0 - eA_0)^2 - \nabla^2 + m^2) \quad (45)$$

The fermions have anti-periodic boundary conditions in the compact time. To regularize the determinant we consider the ratio [41]

$$\Delta(m, eA_0/T) = \frac{\det((-i\partial_0 - eA_0)^2 - \nabla^2 + m^2)}{\det(-\partial_0^2 - \nabla^2 + m^2)} \quad (46)$$

The antiperiodic boundary conditions lead to the expression

$$\Delta(m, eA_0/T) = \prod_{n, \vec{k}} \frac{((2n+1)\pi T - eA_0)^2 + k^2 + m^2}{((2n+1)\pi T)^2 + k^2 + m^2} \quad (47)$$

The product over integers can be evaluated explicitly as [42]

$$\Delta(m, eA_0/T) = \prod_{\vec{k}} \left[1 - \frac{\sin^2(eA_0/2T)}{\cosh^2(\lambda_k/2T)} \right] \equiv \prod_{\vec{k}} \Delta_{\vec{k}}(m, eA_0/T) , \quad (48)$$

where $\lambda_k^2 = \vec{k}^2 + m^2$ are the eigenvalues of the operator $-\nabla^2 + m^2$. In the infinite volume limit the $\prod_{\vec{k}}$ in $\Delta(m, eA_0/T)$ gives rise to an integral on \vec{k} in $\log \Delta$. The result (48) holds in any dimensions, but the integral on \vec{k} can be performed analytically only in two dimensions. In one and three dimensions this integral can only be done for $m = 0$ in which case it gives simple polynomial expressions. In the limit $m = 0$, the effective potentials for A_0 have been discussed in [16]. In two dimensions we obtain

$$\begin{aligned} V(m, eA_0/T) &= \int_{-\infty}^{+\infty} \frac{d^2 \vec{k}}{(2\pi)^2} \log \Delta_{\vec{k}}(m, eA_0/T) = \\ &= -\frac{T^2}{\pi} \left[\frac{m}{T} Li_2(e^{-m/T}, eA_0/T + \pi) + Li_3(e^{-m/T}, eA_0/T + \pi) \right] \end{aligned} \quad (49)$$

where

$$Li_2(r, \theta) = -\int_0^r dx \ln(1 - 2x \cos \theta + x^2)/2x \quad (50)$$

and

$$Li_3(r, \theta) = \int_0^r dx Li_2(x, \theta)/x \quad (51)$$

are the real parts of the dilogarithm and trilogarithm according to the convention of Ref. [43]. As one can see from the definition of $Li_2(r, \theta)$ and $Li_3(r, \theta)$, the effective potential is explicitly invariant, as expected, under the Z symmetry, $eA_0/T \rightarrow eA_0/T + 2\pi n$.

Computing the temporal components of the vacuum polarization function, $\Pi_{00}(p, A_0)$, in an external constant A_0 field and keeping only the term which contributes the leading order in derivatives to the effective action, one obtains

$$Z(m, eA_0/T) = 1 + \frac{e^2}{12\pi m} \left(1 - m \frac{\partial}{\partial m} \right) \frac{\sinh m/T}{\cosh m/T + \cos eA_0/T} \quad (52)$$

We will find it convenient to use the harmonic expansion of the effective potential (49),

$$V(m, eA_0/T) = -\frac{T^2}{\pi} \sum_{n=1}^{\infty} (-1)^n e^{-nm/T} \left(1 + \frac{nm}{T} \right) \frac{\cos(neA_0/T)}{n^3} . \quad (53)$$

There are two limits in which we can obtain more analytic information about the effective potential. In the regime $m \gg T$, T/m and e^2/m are small and e^2/T is unrestricted. The higher harmonics in the effective potential are exponentially small perturbations to the leading term,

$$V(m, eA_0/T) = \frac{Tm}{\pi} e^{-m/T} \cos(eA_0/T) , \quad (54)$$

which is the sine-Gordon potential. The effective field theory in this limit is thus the sine-Gordon model,

$$S_{\text{eff}}^{(m \gg T)}[A_0] = \int d\vec{x} \left\{ \frac{1}{2T} \vec{\nabla} A_0 \cdot \vec{\nabla} A_0 - \frac{Tm}{\pi} e^{-m/T} \cos(eA_0/T) \right\} \quad (55)$$

This model has a phase transition corresponding to the BKT [29, 30] transition in a two-dimensional classical Coulomb gas. The critical behavior associated with this transition has been studied extensively [44, 45, 46, 47, 48, 49]. Wiegmann [47] and Amit et al. [49] showed that in the sine-Gordon model any perturbation of the type $\cos(n\beta\phi)$ to the sine-Gordon potential $\alpha \cos(\beta\phi)/\beta^2$ is irrelevant to the critical behavior of the model. They showed that the scale dimension of $\cos(n\beta\phi)$ is $2n^2$ at the critical point. Consequently for $n > 1$ these operators have scale dimension greater than two and they are indeed irrelevant. Thus we conclude that the critical behaviour in the regime $m \gg T, e^2$ is identical to that of the two dimensional sine-Gordon potential of eq.(54).

The BKT transition occurs at a line of critical points. In the sine-Gordon model with potential

$$\alpha \cos(\beta\phi)/\beta^2$$

the critical line begins at the point

$$(\alpha, \beta^2) = (0, 8\pi) \quad (56)$$

This critical point was originally found by Coleman in his discussion of bosonization and the correspondence of sine-Gordon theory with the massive Thirring model [44]. The line of critical points in the sine-Gordon theory corresponds to a line of critical points for the confinement-deconfinement transition in QED which can be drawn for example in the $(\frac{e^2}{m} e^{-m/T}, e^2/T)$ plane. Comparing eq. (54) with the sine-Gordon potential we see that the QED critical line starts at

$$\left(\frac{e^2}{m} e^{-m/T}, e^2/T\right) = (0, 8\pi) \quad (57)$$

The critical temperature corresponding to this point is

$$T_{\text{crit.}}^{(m \gg T)} = e^2/8\pi \quad (58)$$

The renormalization group was used to study this phase transition, originally by Kosterlitz [30] and Wiegmann [47] and later improved to higher order by Amit and collaborators [49]. The flow diagram is depicted in Fig. 1. There are three regions: The high temperature, deconfined region III, where the model is asymptotically free. This is the case which was analyzed by Coleman [44]. The low temperature region I is confining and has a line of infrared stable fixed points at $m = \infty$, corresponding to $c = 1$ conformal field theory. In Region II, the model is deconfined and is neither asymptotically nor infrared free. The separatrix between regions I and II is the line of BKT phase transitions.

To compute the leading correction to $T_{\text{crit.}}$ due to a finite (but still large) value of the fermion mass, one has first to renormalize the field A_0 , according to

$$A_0^{\text{ren}} \equiv A_0 Z(m, 0)^{1/2} \quad (59)$$

This renormalization changes the sine-Gordon parameter β in the argument of the cosine. The one-loop calculation of the effective action results in the correction

$$T_{\text{crit.}}^{(m \gg T)} = \frac{e^2}{8\pi(1 + e^2/12\pi m + \dots)} \quad (60)$$

As we have shown, the BKT phase transition in QED is a confinement-deconfinement transition. In the spin wave plus Coulomb gas description of the XY-model [40], the BKT phase transition corresponds to a binding-unbinding of vortices. In QED it has the obvious analog of a binding-unbinding transition for charged particle-antiparticle pairs. In the deconfined phase, A_0 fluctuates near one of the minima of the effective potential,

$$\langle A_0 \rangle = 2\pi n T / e \quad (61)$$

In a semiclassical analysis, this expectation value contributes an imaginary chemical potential for the electron. However, this chemical potential can be absorbed by shifting the Matsubara frequency by n units. Thus, in a semiclassical analysis, the thermodynamics in the deconfined phase does not suffer from the difficulties of negative entropy and imaginary thermodynamic potential that affect the meta-stable Z_N phases of QCD [50, 51].

The other limit where the one-loop result is simple is the high temperature limit, where $T \gg m, e^2$. In that limit we must be careful to study the degree of freedom which is periodic ⁶. For this, we define the field

$$a(\vec{x}) \equiv e A_0(\vec{x}) / T \quad (62)$$

so that the effective action for the field $a(\vec{x})$ has the periodicity

$$a(\vec{x}) \rightarrow a(\vec{x}) + 2\pi n \quad (63)$$

The effective action in the high T limit is

$$S_{\text{eff}}^{(T \gg m)}[a] = \int d\vec{x} \left\{ \frac{T}{2e^2} \vec{\nabla} a \cdot \vec{\nabla} a + \frac{T^2}{\pi} \text{Li}_3(1, a + \pi) \right\} \quad (64)$$

Large T is the semi-classical limit for this theory and a must fluctuate near a minimum of the effective potential. In this case, as expected, the Z symmetry is spontaneously broken, corresponding to deconfinement.

3 Maxwell-Chern-Simons theory on the sphere

It is interesting to ask what happens in three-dimensional QED when it is not parity invariant. In this case, the gauge field can have a topological mass term [31] and, naively, one would expect that confinement is not an issue, it is simply absent. In this Section, we shall consider the properties of finite temperature Maxwell-Chern-Simons theory when the space is the 2-sphere. We shall find that there is an analog of the Z symmetry, which exists and has interesting properties even in the absence of matter fields. The symmetry enforces a kind of topological confinement which arises from Dirac's quantization condition for the magnetic field of the monopole. For completeness, we shall also give a careful treatment of the finite temperature path integral in this case. The Minkowski space action for Maxwell-Chern-Simons theory with Wilson-loop sources is

$$S = \int d^3x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \right\} + \sum_i e_i \int_{\Gamma_i} dx^\mu A_\mu \quad (65)$$

⁶ We disagree with the discussion on this point in ref. [16].

The spacetime is a product of time R^1 and S^2 . A similar construction can be carried out where the space is a product of R^1 and any Riemann surface with some additional complications [52].

Since the space is compact and we shall consider the situation where the total charge of the external charge distribution is not zero, the Gauss' law constraint will force us to consider the case where there is a non-zero magnetic charge

$$\int_{S^2} dA = g \neq 0 \quad (66)$$

In this case, the spatial components of the gauge field are not globally defined functions on the sphere but are rather components of a connection on the monopole line bundle. It is well known that, in this situation, extra topological terms are needed to make the Chern-Simons term well-defined [53, 52]. This arises from the fact that the density which one integrates to get the Chern-Simons term is not gauge invariant, but transforms by an exact form under gauge transformations. This makes the integral of the Chern-Simons three-form sensitive to the coordinatization of the monopole line bundle.

This sensitivity can be seen by the following argument. We decompose the gauge field into its spatial and temporal part,

$$A_\mu dx^\mu = A_0 dt + A_i dx^i \equiv A_0 dt + A \quad (67)$$

We construct the monopole line bundle by considering a set of coordinate patches, $\{P_k\}$, which cover the 2-sphere,

$$\cup_k P_k = S^2 \quad (68)$$

and denoting the gauge field in the k 'th patch as A^k . The gauge field in different patches are related by gauge transformations,

$$A^k - A^p = d\chi^{kp} \quad (69)$$

where χ^{kp} is a function which is defined on the intersection region $P^k \cup P^p$. The integral over S^2 is defined using a partition of overlapping patches, so that each point of S^2 is integrated only once. Gauge invariant quantities such as dA or $dAdA$ are not sensitive to details of the choice of coordinate patches. Likewise, the integral of gauge invariant quantities does not depend on the positions of the patches. However, the integral of a non-gauge invariant density, such as the (naive) Chern-Simons term

$$\int dt \sum_k \int_{P_k} \left(A_0 dA^k + A^k dA_0 - A^k \frac{d}{dt} A^k \right) \quad (70)$$

depends on the position of the patches. For example, the contribution to Gauss' law density arising from the Chern-Simons term is obtained by taking a variational derivative of the Chern-Simons term by A_0 . To do this, it is necessary to translate $A_0 \rightarrow A_0 + \delta A_0$, to isolate δA_0 by integrating by parts and then to identify the functional derivative as the coefficient of δA_0 under the integral. With this procedure one obtains the charge density

$$J_0 = 2dA + \sum_{kp} \int_{P_k \cap P_p} \delta(x - P_k \cap P_p) d\chi^{kp} \quad (71)$$

This has the unappealing feature that some charge lives on the arbitrarily chosen transition regions (in fact, on the arbitrarily chosen boundaries between patches which are inside the transition regions). The surface terms which must be added to the Chern-Simons term cure this difficulty. In the present case, they are integrated on the intersection regions of coordinate patches and cancel the terms obtained when the second term in the above naive Chern-Simons term is integrated by parts to obtain the expression

$$\int dt \sum_k \int_{P_k} (2A_0 dA^k - A^k \dot{A}^k) \quad (72)$$

This version of the Chern-Simons term will be sufficient for our purposes in the following.

3.1 Canonical quantization

We shall examine the electrostatics of an array of classical charges. In this case, the source term in the action has the form

$$\sum_i e_i \int dt A_0(x_i, t) \quad (73)$$

In particular, we are interested in the free energy of this system as a function of particle positions. We shall not require global neutrality of the charge distribution. However, the consistency of the monopole bundle will force us to use charges which are integer multiples of a basic charge, e , compatible with the Dirac quantization condition.

If the total charge is non-zero, the gauge constraint, which is obtained by taking a functional derivative of the action (65) by A_0 is

$$\nabla \cdot E + \kappa B + \sum_i e_i \delta(x - x_i) \sim 0 \quad (74)$$

The electric field is gauge invariant and must be a globally defined vector field on S^2 . Therefore, the integral of the divergence of the electric field over the space S^2 must vanish. The integral of Gauss' law then implies that, when the total electric charge is not zero, there is also a non-zero magnetic flux,

$$\kappa g + \sum_i e_i = 0 \quad (75)$$

It is convenient to separate the effect of the magnetic background field by decomposing the gauge field into a classical time-independent part containing the monopole field and a time dependent part with no overall magnetic flux and which is allowed to have quantum fluctuations,

$$A_i(x, t) \rightarrow A_{M,i}(x) + A_i(x, t) \quad (76)$$

In eq.(76)

$$\int_{S^2} \nabla \times A_M = g \quad (77)$$

and A_M is defined in such a way that the classical magnetic field is constant,

$$B_M = \nabla \times A_M = g/4\pi R^2 \quad (78)$$

(with R the radius of S^2) so that

$$\int B = 0$$

and

$$\int B_M B = 0 \quad .$$

Substituting (76) into the action yields

$$S = \int dt \int_{S^2} \left\{ \frac{1}{2} (\dot{A}_i - \nabla_i A_0)^2 - \frac{1}{2} (B_M^2 + B^2) + \kappa A_0 B_M + \kappa A_0 B - \frac{\kappa}{2} A \times \dot{A} - \frac{\kappa}{2} \frac{d}{dt} (A_M \times A) \right\} + \sum_i \int dt e_i A_0(t, \vec{x}_i) \quad (79)$$

where, now all variables are single-valued, globally defined functions on S^2 and the monopole gauge field B_M is a classical variable. The only remaining multi-valued term $\int \frac{d}{dt} (A_M \times A)$ is a total time derivative term and therefore is not important for the canonical quantization which we shall do in the following.

Canonical quantization proceeds by identifying the canonical momenta

$$\Pi_0 \sim 0 \quad (80)$$

$$\Pi_i = \dot{A}_i - \nabla_i A_0 + \frac{\kappa}{2} \epsilon_{ij} A_j \quad (81)$$

The first relation is a primary constraint. The Hamiltonian is

$$H = \int_{S^2} \left\{ \frac{1}{2} \left(\Pi_i - \frac{\kappa}{2} \epsilon_{ij} A_j \right)^2 + \frac{1}{2} B^2 + \frac{1}{2} B_M^2 \right\} \quad (82)$$

and the canonical commutation relations are given by

$$[A_i(\vec{x}), \Pi_j(\vec{y})] = i \delta_{ij} \delta(\vec{x} - \vec{y}) \quad (83)$$

$$[A_0(\vec{x}), \Pi_0(\vec{y})] = i \delta(\vec{x} - \vec{y}) \quad (84)$$

The gauge constraint arises as a secondary constraint from requiring that the primary constraint $\Pi_0 \sim 0$ is time-independent

$$G(\vec{x}) \equiv \vec{\nabla} \cdot \vec{\Pi}(\vec{x}) + \frac{\kappa}{2} B(\vec{x}) + \kappa B_M + \sum_i e_i \delta(\vec{x} - \vec{x}_i) \sim 0 \quad (85)$$

The operator $G(\vec{x})$ generates time-independent gauge transformations and commutes with the Hamiltonian which is invariant under the gauge transformation

$$\begin{aligned} \vec{A}(\vec{x}) &\rightarrow \vec{A}(\vec{x}) + \left\{ \vec{A}(\vec{x}), \int d\vec{y} \chi(\vec{y}) G(\vec{y}) \right\} = \vec{A}(\vec{x}) - \vec{\nabla} \chi(\vec{x}) \\ \vec{\Pi}(\vec{x}) &\rightarrow \vec{\Pi}(\vec{x}) + \left\{ \vec{\Pi}(\vec{x}), \int d\vec{y} \chi(\vec{y}) G(\vec{y}) \right\} = \vec{\Pi}(\vec{x}) - \frac{\kappa}{2} \vec{\nabla}^* \chi(\vec{x}) \end{aligned} \quad (86)$$

where $\vec{\nabla}_i^* \equiv \epsilon_{ij} \nabla_j$.

The dynamical system with Hamiltonian (82) and canonical commutator (83) is internally consistent and can be quantized as it is (with the subtlety

that the ground state is not normalizable). The primary constraint, (80) is solved by imposing the auxiliary gauge fixing condition

$$A_0(\vec{x}) \sim 0 \quad (87)$$

and thereby eliminating both A_0 and Π_0 . The Gauss' law constraint (85) must then be imposed as a physical state condition. Since the operator $G(\vec{x})$ commutes with the Hamiltonian, it can be diagonalized simultaneously with the Hamiltonian. Then the simultaneous eigenstates of the Hamiltonian and $G(\vec{x})$ which are in the kernel of $G(\vec{x})$ are chosen as the physical states,

$$G(x)| \text{ physical state } \rangle = 0 \quad . \quad (88)$$

We can form a projection operator onto physical states by considering the set of all gauge transforms. Formally,

$$\mathcal{P} = \frac{1}{\text{vol.G}} \int [d\chi(x)] \exp \left(i \int_{S^2} \chi(x) G(x) \right) \quad . \quad (89)$$

This operator satisfies the property

$$\mathcal{P}^2 = \mathcal{P}$$

(This is a formal statement due to the infinite volume of the gauge group. This is the same difficulty which appears in the normalization of the states.)

The form of Gauss' law indicates that the gauge symmetry is realized in a projective representation. For example, if we represent the canonical commutation relation in the functional Schrödinger picture, where states are wave-functionals $\Psi[\vec{A}]$ of the classical field \vec{A} , and the canonical momentum is a functional derivative operator,

$$\Pi_i(\vec{x}) = \frac{1}{i} \frac{\delta}{\delta A_i(\vec{x})} \quad . \quad (90)$$

Then a physical state which obeys the gauge condition

$$G(\vec{x})\Psi_{\text{phys}}[\vec{A}] = 0 \quad (91)$$

gauge transforms as

$$\Psi_{\text{phys}}[\vec{A} - \vec{\nabla}\chi] = e^{i\kappa \int \chi(B_M + B/2) + i \sum_i e_i \chi(\vec{x}_i)} \Psi_{\text{phys}}[\vec{A}] \quad (92)$$

In the next subsection, we shall find the functional integral representation of the thermodynamic partition function.

3.2 Functional integral representation of the partition function

In this subsection, we shall discuss the derivation of the functional integral representation of the thermodynamic partition function. It is obtained by taking the trace over physical states of the Gibbs distribution operator

$$\rho = e^{-H/T} \quad (93)$$

where H is the Hamiltonian and T is the temperature. (We work in a system of units where the Boltzmann constant, the speed of light and Planck's constant are equal to one.) We shall consider the unconstrained space of states which represent the canonical commutation relation (83) and insert into the trace a projection operator which projects onto the physical states:

$$Z[T] = \sum_s \langle s | e^{-H/T} \mathcal{P} | s \rangle \quad (94)$$

Explicitly, this can be written as

$$Z[T] = \frac{1}{\text{vol } G} \int [d\chi(\vec{x})] \int [da_i(\vec{x})] \langle a | e^{-H/T} e^{i\kappa \int \chi(B_M+B/2)+i \sum e_i \chi(\vec{x}_i)} | a + d\chi \rangle \quad (95)$$

where we have taken the trace using the eigenvectors of the “position” operator \vec{A} ,

$$A_i(\vec{x})|a\rangle = a_i(\vec{x})|a\rangle \quad (96)$$

The integration over χ in (95) projects the trace onto gauge invariant states. Since the Hamiltonian is gauge invariant, it is sufficient to insert the projection operator once. The field $\chi(\vec{x})$ is proportional to the temporal component of the gauge field, $A_0(\vec{x})$, in the time-independent gauge

$$\chi(\vec{x}) \equiv A_0(\vec{x})/T \quad (97)$$

We are particularly interested in deriving an effective action for the gauge function $A_0(\vec{x})$. It is defined by

$$e^{-S_{\text{eff}}[A_0]} \equiv \sum_{B_M} \int [d\vec{a}] \langle \vec{a} | e^{-H/T} e^{i\frac{\kappa}{T} \int A_0(B_M+B/2)} | \vec{a} + \vec{\nabla} A_0/T \rangle \quad (98)$$

where we have omitted the external charges and we have summed over magnetic monopole number. The subsequent integration over $A_0(\vec{x})$, which is needed in order to obtain the partition function of the system in the presence of external charges, will enforce Gauss' law. In particular it will project onto the sector with the correct magnetic charge.

The partition function of the system in the presence of external charges is the correlator

$$Z[T, (e_i, \vec{x}_i)] = \frac{\int [dA_0(\vec{x})] e^{-S_{\text{eff}}[A_0]} \prod_j e^{ie_j A_0(\vec{x}_j)/T}}{\int [dA_0(\vec{x})] e^{-S_{\text{eff}}[A_0]}} \quad (99)$$

The matrix element in (95) has the standard phase space path integral expression [54], so that

$$e^{-S_{\text{eff}}[A_0]} \equiv \int [d\vec{a}][d\vec{\pi}] e^{\int_0^{1/T} d\tau \int_{S^2} \left\{ i\vec{\pi} \cdot \dot{\vec{a}} - \frac{1}{2}(\pi_i - \frac{\kappa}{2}\epsilon_{ij}a_j)^2 - \frac{1}{2}(B_M^2 + b^2) \right\} + i\frac{\kappa}{T} \int_{S^2} A_0(B_M + b/2)} \quad (100)$$

where the time interval is $\tau \in [0, 1/T]$, the spatial integral in the action is taken over S^2 , the canonical momentum π has open boundary conditions and the gauge field has the boundary condition which is periodic up to a twist by a gauge transformation,

$$\vec{a}(1/T, \vec{x}) = \vec{a}(0, \vec{x}) - \vec{\nabla} A_0(\vec{x})/T \quad (101)$$

We have denoted the fluctuating part of the magnetic field as $b = \vec{\nabla} \times \vec{a}$. Equation (100) gives the effective action up to an overall temperature dependent but A_0 independent constant.

The canonical momentum can be integrated in order to present the functional integral in configuration space, up to an irrelevant temperature dependent factor,

$$e^{-S_{\text{eff}}[A_0]} = \sum_{B_M} \int [d\vec{a}] e^{-\int_0^{1/T} d\tau \int_{S^2} \left(\frac{1}{2} \dot{\vec{a}}^2 + \frac{1}{2} (B_M^2 + b^2) + i \frac{\kappa}{2} \vec{a} \times \dot{\vec{a}} \right)} \cdot e^{i \frac{\kappa}{T} \int_{S^2} A_0 (B_M + b/2)} \quad (102)$$

It is convenient to untwist the boundary condition using the change of variables,

$$\vec{a}(\tau, \vec{x}) \rightarrow \vec{a}(\tau, \vec{x}) - \tau \vec{\nabla} A_0(\vec{x}) \quad (103)$$

with the result that

$$e^{-S_{\text{eff}}[A_0]} = \sum_{B_M} e^{-\frac{1}{2T} \int_{S^2} (\nabla A_0)^2 + i\kappa \int A_0 B_M / T} \cdot \int [d\vec{a}] e^{-\int_0^{1/T} d\tau \int_{S^2} \left(\frac{1}{2} \dot{\vec{a}}^2 + \frac{1}{2} B_M^2 + \frac{1}{2} b^2 + i \frac{\kappa}{2} \vec{a} \times \dot{\vec{a}} - i\kappa A_0 b \right)} \quad (104)$$

The Gaussian integral over \vec{a}_i can now easily be done. If we choose B_M to be a constant magnetic field on the sphere, the integral over $\vec{a}(\vec{x})$ in (104) yields

$$e^{-S_{\text{eff}}[A_0]} = \sum_{B_M} \exp \left(-\frac{g^2}{2T(4\pi R^2)} + i \frac{\kappa g}{(4\pi R^2)T} \int_{S^2} A_0 - \frac{1}{2T} \sum_{l,m;l \neq 0} |A_0(l,m)|^2 (l(l+1)/R^2 + \kappa^2) \right) \quad (105)$$

where we have dropped an irrelevant infinite constant, recalled the fact that $\int_{S^2} B_M = g = 4\pi R^2 B_M$ and we have expanded $A_0(\vec{x})$ in spherical harmonics,

$$A_0(\vec{x}) = \sum_{lm} A_0(l,m) Y_{lm}(\vec{x}) \quad (106)$$

$l \geq 0$ are integers of the usual angular momentum spectrum, the spectrum of the laplacian $-\nabla^2$ being $l(l+1)R^2$, and $m = -l, -l+1, \dots, l$ for each l . Notice that the non-zero modes of the gauge function $\chi(\vec{x})$ are governed by a massive euclidean free field theory. The zero modes, on the other hand are coupled to the monopole moments which must be summed.

If we recall the monopole quantization condition, $g = 2\pi n/e$, the summation over monopoles is

$$\sum_n \exp \left(-\frac{(2\pi)^2}{(4\pi R^2)2e^2 T} n^2 + \frac{2\pi i\kappa}{eT} n \hat{A}_0 \right) \quad (107)$$

where

$$\hat{A}_0 \equiv \frac{1}{4\pi R^2} \int_{S^2} A_0$$

Using the Poisson resummation formula, the sum can be presented as

$$\sqrt{\frac{(4\pi R^2)e^2 T}{2\pi}} \sum_n \exp \left(-\frac{(4\pi R^2)e^2 T}{2} \left(\frac{\kappa}{eT} \hat{A}_0 + n \right)^2 \right) \quad (108)$$

which explicitly exhibits periodicity in \hat{A}_0 ,

$$\hat{A}_0 \rightarrow \hat{A}_0 + \frac{eT}{\kappa} \quad (109)$$

The summation over monopoles effectively represents this symmetry in the Villain form [55],

$$e^{-S_{\text{eff}}[A_0]} = \sum_n \exp \left(-\frac{1}{2T} \int_{S^2} \left\{ \vec{\nabla} A_0 \cdot \vec{\nabla} A_0 + \kappa^2 \left(A_0 + \frac{eT}{\kappa} n \right)^2 \right\} \right) \quad (110)$$

This action is identical (although one dimension higher) to the action that was found for the Schwinger model in ref. [19]. In that case, we argued that the Z symmetry is always broken in the infinite volume limit. This symmetry breaking is interpreted as screening, similar to that which occurs in a Higgs phase, rather than deconfinement. We conjecture that a topologically massive gauge theory screens, rather than deconfines. We shall derive support for this conjecture from the variational calculation in Section IV where we find indications of a rapid change of the behavior of the system between what we here interpret as a screening phase and what we would properly call a deconfined phase.

In the next subsection we shall examine the consequences of this symmetry of the effective action (110) and we will show that it is also spontaneously broken in the infinite volume limit.

3.3 Spontaneous breaking of Z symmetry

The Polyakov loop operator transforms under (109) as

$$e^{ineA_0(\vec{x})/T} \rightarrow e^{ine^2/\kappa} e^{ineA_0(\vec{x})/T} = e^{2\pi in \cdot p/q} e^{ineA_0(\vec{x})/T} \quad (111)$$

where we have defined (as in (34))

$$\kappa = \frac{e^2}{2\pi} \frac{p}{q}$$

This symmetry has implications for the correlator of Polyakov loop operators,

$$Z[T, (e_i, x_i)] = \left\langle \prod_j e^{ie_i A_0(\vec{x}_j)/T} \right\rangle \quad (112)$$

which depend on the ratio p/q .

- p/q is irrational. Then the correlator (112) vanishes unless $\sum_i e_i = 0$.
- p/q is rational. Then the correlator (112) vanishes unless $\sum_j e_j = e \cdot \text{integer} \cdot q$. Since the consistency with the monopole bundle requires that the charges are quantized in units of e , so $\sum_j e_j = \text{integer} \cdot e$, the Z symmetry here is actually a subgroup of Z , Z_q , the additive group of the integers modulo q .
- If p is an integer and $q = 1$ the correlator (112) is unrestricted.

It is interesting to observe that only charges which are integer multiples of qe are allowed on the sphere. This is no surprise as it is a direct consequence of Gauss' law. The integral of Gauss' law together with the Dirac quantization condition yields the constraint $\text{charge} = qe \cdot \text{integer}$. Thus, only charges which are integer multiples of q are consistent with the Gauss' law constraint on the sphere. The Z_q symmetry enforces this "topological constraint".

The Z_q -symmetry is invariably broken in the infinite volume limit. To see this, we consider the two-point correlator of loop operators

$$\left\langle e^{ieA_0(\vec{x})/T} e^{-ieA_0(\vec{0})/T} \right\rangle = \text{const.} \cdot \exp \left(-\frac{e^2}{2T} (\vec{x} | \frac{1}{-\vec{\nabla}^2 + \kappa^2} | \vec{0}) \right) \quad (113)$$

If we take the limit as the volume goes to infinity, followed by the limit as the separation of the points in the correlator goes to infinity, the two-point function approaches a non-zero constant,

$$\lim_{|\vec{x}| \rightarrow \infty} \lim_{R^2 \rightarrow \infty} \left\langle e^{ieA_0(\vec{x})/T} e^{-ieA_0(\vec{0})/T} \right\rangle = \text{const.} \quad (114)$$

This implies that the Z_q symmetry is spontaneously broken in the infinite volume limit.

This leaves us with the correct conclusion that the topologically massive gauge theory is not a confining theory. In fact its electrostatic interactions are short-ranged and Yukawa-like. Their large distance fall-off is governed by the inverse topological mass.

Thus, for all practical purposes, the topological mass of the photon contributes a mass term to the effective action for A_0 . The fact that this mass term is periodic when the volume is finite is irrelevant in the infinite volume limit. If we couple topologically massive QED to matter fields, we would expect that, in the limit where the matter field masses are large, the effective field theory is the massive sine-Gordon theory,

$$S_{\text{eff}}^{m>>T}[A_0] = \int d^2x \left\{ \frac{1}{2T} \vec{\nabla} A_0 \cdot \vec{\nabla} A_0 + \frac{\kappa^2}{2T} A_0^2 - \frac{mT}{\pi} e^{-m/T} \cos(eA_0/T) \right\} \quad (115)$$

In the next Section, we shall study this model using a variational approach.

4 Variational approach

In this section, we shall discuss a variational approach to the problem of showing the existence of a phase transition in the one-loop effective theory. We have argued that the effective field theory where the phase transition can be studied is the sine-Gordon theory with an additional mass term for the boson,

$$S_{\text{eff}} = \int d^2x \left\{ \frac{1}{2} \nabla \phi \cdot \nabla \phi + \frac{\mu^2}{2} \phi^2 - \frac{\alpha}{\beta^2} \cos \beta \phi \right\} \quad (116)$$

Here, the mass term for the boson is the topological photon mass, $\mu = \kappa$. Also, in the limit where the fermion masses of QED are much greater than the temperature and charge squared, we have $\beta = e/\sqrt{T}$ and $\alpha = e^2 m e^{-m/T}/\pi$. The ultraviolet cutoff is the fermion mass, m .

As we have discussed in the previous sections, the mass term for the boson should really be a periodic one. However, quantum effects invariably break the translation symmetry for such a Bose field in the limit where the volume is infinite. We shall therefore ignore the translation symmetry.

In Ref. [56], the behavior of the model (116) was investigated around the point $\beta^2 = 4\pi$, and a phase transition at the corresponding critical temperature $\tilde{T}_c = e^2/4\pi$ was suggested. In subsequent work on the massless model [49], it was however found that the divergencies at $\beta^2 = 4\pi$ can be handled within the renormalization scheme, and the behavior is smooth. Accordingly, we shall concentrate on studying the model only in the neighborhood of $\beta^2 = 8\pi$.

In Section II we reviewed the argument of ref. [17] that, when the topological mass is zero, there is a Berezinskii-Kosterlitz-Thouless confinement-deconfinement transition at the temperature $T_c[e^2, m] = e^2/8\pi(1 + \dots \mathcal{O}(e^2/m))$, corresponding to $\beta^2 \simeq 8\pi$ for the sine-Gordon parameter. When there is a topological mass, the system always fails to attain its asymptotic scaling regime, and the flow diagram cannot be that of a BKT transition. In fact, at a scale of the order of the inverse topological mass there is a crossover from a behavior governed by the long ranged Coulomb interaction to one governed by a finite range Yukawa interaction. There is no confinement in the strict sense, since the Coulomb interaction cuts off at a given distance. However, when the topological mass is small, much smaller than any of the other dimensional parameters (particularly the confining scale which is governed by e^2), for low temperature and spatial scales much less than the inverse topological mass and of the order of the confinement scale, the physical behavior of the system should be much like a confining one with a confining interaction between oppositely charged particles. One expects that, at or near the confinement-deconfinement temperature, a drastic change in the properties of the system takes place, though not a true phase transition. Such a rapid change is between a quasi-confining and a de-confined behavior with the electric mass of the photon moving from the value of the topological mass, in the low temperature quasi-confined phase, to the value of the much larger Debye screening mass, in the de-confined phase.

If one imagines separating a test particle-antiparticle pair at short distances the potential energy should increase with distance just as it does in a strictly confining system. If this increasing energy gets large enough it can produce a pair of dynamical charges which partially screen the charges of the source. Thus, at short distances, separating a particle-antiparticle pair should produce mesons, rather than free charged particles. However, at distances larger than the inverse of the topological mass, charged particles should behave as free particles. We shall interpret this as a screening (as distinct from deconfined) phase analogous to what happens in the Schwinger model or massless two-dimensional QCD.

If we then increase the topological mass to the confining scale, the system should go continuously to one which deconfines at all scales. We shall qualitatively study these behaviours by means of a variational approach in the next subsections.

4.1 Jensen's inequality

In statistical mechanics, a variational approach uses Jensen's inequality. First, we shall give a brief review of this inequality and its derivation. Consider a statistical system with Hamiltonian H_1 which we want to study the statistical mechanics of, but are unable to solve for the sum over states to obtain the partition function or the correlators exactly. We consider another test Hamiltonian, H_0 which contains some parameters and with which we can solve for the partition function and correlation functions of observables analytically. Then, consider the Hamiltonian which interpolates linearly between them,

$$H_\lambda = \lambda H_1 + (1 - \lambda) H_0 \quad (117)$$

and the free energy

$$W(\lambda) = -\ln \sum_s e^{-H_\lambda(s)} \quad (118)$$

It has the properties

$$\begin{aligned} \frac{\partial W(\lambda)}{\partial \lambda} &= \langle H_1 - H_0 \rangle_\lambda \\ \frac{\partial^2 W(\lambda)}{\partial \lambda^2} &= -\langle (H_1 - H_0)^2 \rangle_\lambda + \langle H_1 - H_0 \rangle_\lambda^2 \leq 0 \end{aligned} \quad (119)$$

where $\langle \dots \rangle_\lambda$ is the expectation value in the ensemble with Hamiltonian H_λ . Since the curvature of $W(\lambda)$ as a function of λ is always less than or equal to zero, $W(\lambda)$ obeys the inequality

$$W(\lambda) \leq W(0) + \lambda \cdot \left(\frac{\partial W(\lambda)}{\partial \lambda} \right)_{\lambda=0} \quad (120)$$

which, evaluated at $\lambda = 1$ is Jensen's inequality

$$W(1) \leq W(0) + \langle H_1 - H_0 \rangle_0 \quad (121)$$

This establishes an upper bound on the free energy of the system of interest by the system in the variational ansatz which can be optimized by adjusting the parameters of the variational ansatz. The bound is saturated only when the ensembles are identical ($H_1 = H_0$).

4.2 Variational computations

We are faced with the problem of computing the thermodynamics of the system with Hamiltonian function

$$H_1 = \int d^2x \left\{ \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \frac{\mu^2}{2} \phi^2 - \frac{\alpha}{\beta^2} \cos \beta \phi \right\} \quad (122)$$

We shall study this system variationally by beginning with the test ensemble governed by the quadratic Hamiltonian

$$H_0 = \int d^2x d^2y \frac{1}{2} \phi(\vec{x}) \xi((\vec{x} - y)^2) \phi(y) \quad (123)$$

with an ultraviolet cutoff Λ , which is of order the mass of the matter field in the original model. The Gaussian functional integrals appearing on the right-hand side of Jensen's inequality in this case can easily be done with result

$$\frac{1}{V}W \leq \frac{1}{2} \int^\Lambda \left\{ \ln \left(\frac{\xi(p^2)}{\Lambda^2} \right) + \frac{p^2 + \mu^2}{\xi(p^2)} - 1 \right\} - \frac{\alpha}{\beta^2} e^{-\frac{\beta^2}{2} \int^\Lambda \xi^{-1}} . \quad (124)$$

Here the measure for the cutoff momentum integration is

$$\int^\Lambda \equiv \int \frac{d^2 p}{(2\pi)^2} \theta(\Lambda^2 - p^2) \quad (125)$$

and has been extracted a factor of the spatial volume to get the free energy density.

In order to optimize the ansatz, we take the variational derivative of the right hand side to obtain

$$0 = \xi(p^2)^{-2} \left(\xi(p^2) - p^2 - \mu^2 - \alpha e^{-\frac{\beta^2}{2} \int^\Lambda \xi^{-1}} \right) \quad (126)$$

The equation for the minimum is solved by

$$\xi(p^2) = p^2 + M^2 \quad (127)$$

where the mass parameter M^2 satisfies the equation

$$M^2 = \mu^2 + \alpha \left(\frac{M^2/\Lambda^2}{1 + M^2/\Lambda^2} \right)^{\beta^2/8\pi} \quad (128)$$

We substitute (127) into (124) and integrate to obtain

$$\frac{1}{V}W \leq \frac{1}{8\pi} \left(\Lambda^2 \ln \left(\frac{\Lambda^2 + M^2}{\Lambda^2} \right) + \mu^2 \ln \left(\frac{\Lambda^2 + M^2}{M^2} \right) - 8\pi \frac{\alpha}{\beta^2} \left(\frac{M^2}{\Lambda^2 + M^2} \right)^{\beta^2/8\pi} \right) \quad (129)$$

The regime that we are interested in is where the cutoff, (the electron mass) is very large and the other dimensional parameters μ and α are very small. We look for a minimum of the potential (129) where M is of the same order of magnitude as μ and α . This corresponds to seeking a solution of sine-Gordon theory which is consistently renormalized as a relativistic quantum field theory.

4.2.1 Sine-Gordon theory ($\mu^2 = 0$, $\Lambda \rightarrow \infty$)

It is interesting to explore the minima of (129) in the case of pure sine-Gordon theory, when $\mu^2 = 0$. This can be done using a slightly different variational method by Coleman [44].

We must seek a solution of the equation for the variational mass where $M^2 \ll \Lambda^2$. To do this, we must first renormalize the bare coefficient of the cosine term in the action,

$$\alpha \equiv \alpha_R \left(\frac{\Lambda^2}{a^2} \right)^{\beta^2/8\pi} \quad (130)$$

where a is an arbitrary mass scale which accompanies renormalization. Then, up to an infinite, M independent constant, the variational free energy is

$$\frac{1}{V}W \leq \frac{1}{8\pi} \left(M^2 - 8\pi \frac{\alpha_R}{\beta^2} \left(\frac{M^2}{a^2} \right)^{\beta^2/8\pi} \right) \quad (131)$$

Clearly, when $\beta^2/8\pi < 1$, the point $M = 0$ is a local maximum of the potential. There is a minimum at $M \neq 0$, determined by the finite dimensional parameters a and α_R and by β . On the other hand, when $\beta^2/8\pi > 1$, there is a local minimum of the potential at $M = 0$ but for large M , it is unbounded from below. This means that $M = 0$ is an unstable state and the value of M at the true minimum of the potential is of order the (infinite) cutoff.

Thus, we regain Coleman's result [44]. Without the renormalization of the β^2 parameter, which was shown to be necessary in ref. [49], the pure sine-Gordon model has a phase transition at the point $\beta_c^2 = 8\pi$ from a phase where the theory is approximately solved by a boson with finite mass to one where the boson has infinite mass, of order of the cutoff.

4.2.2 Sine-Gordon theory: parity invariant QED ($\mu^2 = 0$, Λ finite)

Let us now analyze the case in which the cutoff is finite and of the order of the electron mass m . We can then analyze directly the potential (129) with $\Lambda \sim m$ and $\mu = 0$. It can be easily seen that, independently on the value of Λ , the potential (129) for $\beta^2/8\pi < 1$ behaves as before, *i.e.* at the point $M = 0$ has a local maximum and an absolute minimum at $M \neq 0$. For $\beta^2/8\pi > 1$, however the minimum at $M = 0$ is now an absolute minimum for any $M < \Lambda$, the potential is always bounded below. $M = 0$ describes a stable state, so that the transition is between a phase characterized by the behavior of a two-dimensional massless boson to one characterized by a massive boson.

This can be interpreted as a change in the symmetry, since a massless boson has the field translation symmetry, $\phi(\vec{x}) \rightarrow \phi(\vec{x}) + \text{const.}$ whereas a massive boson does not. If we translate the parameters of the sine-Gordon theory into those of the effective action for QED (as in the discussion after equation (116)), the phase transition occurs at the critical temperature $T_c = e^2/8\pi$. Above this temperature, the boson has a mass, corresponding to deconfined phase which screens electric charges by virtue of the "Debye" mass M (region III in Fig.1), whereas below this temperature the boson is massless (region I in Fig.1), corresponding to a confining phase which cannot screen the long ranged electric fields of incommensurate charges.

4.2.3 Sine-Gordon theory with a mass ($\mu^2 \neq 0$, $\Lambda \rightarrow \infty$)

Let us now consider the case of massive sine-Gordon theory with $\mu^2 > 0$. The variational potential is made finite by the same renormalization of the parameter α as in the massless case. The variational free energy is then

$$\frac{1}{V}W = \frac{1}{8\pi} \left(M^2 - \mu^2 \ln(M^2/a^2) - 8\pi \frac{\alpha_R}{\beta^2} \left(\frac{M^2}{a^2} \right)^{\beta^2/8\pi} \right) \quad (132)$$

If $\beta^2 < 8\pi$, the minimum of the potential occurs at a finite value of M^2 . This value depends crucially also on the topological mass μ and for μ very small

occurs approximately at the same value of the $\mu = 0$ case, for μ sufficiently large it occurs at μ . As in massless sine-Gordon theory, without a renormalization of β^2 , the potential is unbounded below if $\beta^2 > 8\pi$, independently on the value of μ . This implies that the global minimum of the potential is of order the ultraviolet cutoff.

4.2.4 Sine-Gordon theory with a mass: topologically massive QED ($\mu^2 \neq 0$, Λ finite)

For μ very small and $\beta^2 < 8\pi$ (high temperature) the potential (129) has an absolute minimum at a finite value of M much larger than μ , independently on the value of the cutoff Λ . For $\beta^2 > 8\pi$ (low temperature) the absolute minimum rapidly moves to the small value $M \simeq \mu$, so that the transition is between two distinct massive behaviors. For a very tiny μ the crossover in the BKT should arise only at very large scales. The behavior of the system in the two phases should be quite different. In the low temperature region all the components of the gauge field have a small mass (the topological mass), and this phase should be very much like the Higgs phase of the Schwinger model [19]. In the high temperature region, the electric mass grows to the much bigger Debye mass. This region is essentially similar to the plasma phase of the parity invariant theory. Moreover, the Debye screening mass is temperature dependent, whereas the topological mass does not depend on temperature.

It is interesting that this transition occurs even when there is an explicit mass term in the action, provided that the mass term is sufficiently small. Although this transition is in a sense associated with vortex binding-unbinding, just as it is in the sine-Gordon theory which describes the Coulomb gas, it is not a confinement-deconfinement transition in the strict sense, since the Z symmetry is broken in both phases.

When the topological mass μ is large, the minimum of the potential is always at μ regardless of the value of β^2 , so that the crossover arises at short scales ($\sim 1/\mu$) and the flow diagram is completely destroyed. No drastic change in the behavior of the system should be observed as the temperature is changed in the neighborhood of $T_c = e^2/8\pi$.

5 Discussion

In this paper, we have shown that deconfinement in finite temperature QED can be characterized as breaking of a certain global discrete symmetry. We have also shown that a confinement-deconfinement transition takes place in parity invariant 2+1-dimensional QED, at least in the regime where the electron mass is large.

In a sense, the latter fact is no surprise. When the electromagnetic coupling e^2 is small compared to the electron mass m so that vacuum fluctuations are suppressed, and when the temperature is also smaller than the electron mass, the thermal state is to a good approximation a dilute two-dimensional neutral Coulomb gas of thermally excited electrons and positrons. It is well known that this Coulomb gas, even when very dilute, has a BKT transition. Below the transition temperature, the electrons and positrons are bound

into pairs. Above the transition temperature, electrons and positrons are approximately free particles. One physical prediction which can be deduced from the presence of the BKT transition is the universal property of the phase transition associated with the bulk modulus of spin waves in the gapless, Kosterlitz-Thouless phase. The exponents in the correlators of Polyakov loop operators in that phase are determined by a single correlation length divided by the temperature.

There is a similar picture of the topologically massive theory. If the topological mass is small, at weak coupling and temperatures somewhat less than the electron mass, the thermal state is to a good approximation that of the dilute Yukawa gas. There is a rapid change between a quasi-confining and a deconfined behavior close to the BKT critical point of the parity invariant theory.

In both cases, our analysis is only valid where the electron mass is large. We expect that the phase transition, or more correctly the BKT line of phase transitions, persists for some time as we lower the electron mass or raise the electric charge. However, the resulting strong coupling regime is out of the domain of validity of our analysis.

An interesting, experimentally testable consequence of the breaking of the Z symmetry is the existence of domain walls. For the existence of such, it is enough to have field configurations that are non-contractable loops in the $U(1)$ group. The question of domain walls has been addressed in gluodynamics [33, 34] and massless QED [16]. Whether they correspond to real Minkowski space objects, is an interesting open question. It is instructive to note that upon refermionizing our effective sine-Gordon model (55) along the lines of [44], domain walls in sine-Gordon theory correspond to worldlines of fermions in the resulting Thirring model.

Phase transitions analogous to the one which we conjecture to exist in topologically massive QED have been studied experimentally in quasi two-dimensional condensed matter systems, particularly charged vortex arrays in superconducting films [57] and have also been useful in theoretical work on high T_C superconductivity [58] where a type of “smoothed” BKT transition is discussed. The experimental study of analog systems in condensed matter physics could help to resolve some of the theoretical questions raised by our current work.

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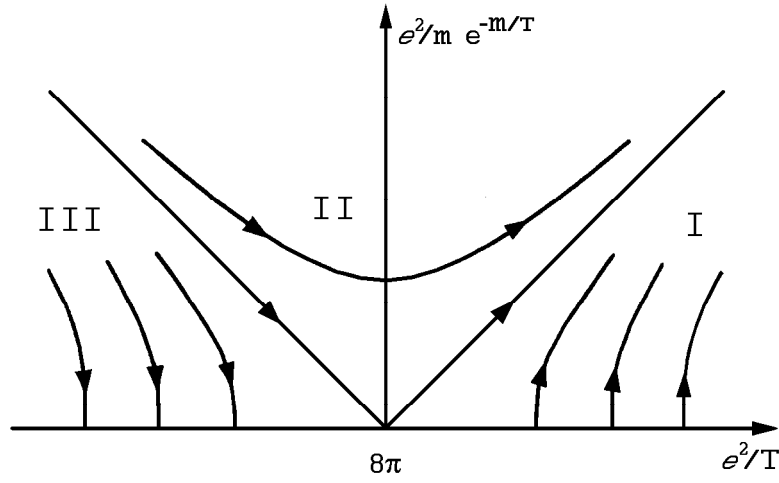


Figure 1.

Renormalization flow diagram for the BKT transition. The arrows denote flow toward the ultraviolet. Region I is the confining phase whereas regions II and III are deconfined. Region III is asymptotically free whereas in region I there is a line of infrared stable fixed points which represent $c = 1$ conformal field theories. The separatrix between regions I and II is the line of BKT phase transitions. The critical behaviour of the system at the latter phase transition is that of an $SU(2)$ Wess-Zumino-Witten model.